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Sequences of Infinite Bifurcations and Turbulence in a Five-Mode Truncation of the Navier–Stokes Equations

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Two infinite sequences of orbits leading to turbulence in a five-mode truncation of the Navier–Stokes equations for a 2-dimensional incompressible fluid on a torus are studied in detail. Their compatibility with Feigenbaum's theory of universality in certain infinite sequences of bifurcations is verified and some considerations on their asymptotic behavior are inferred. An analysis of the Poincaré map is performed, showing how the turbulent behavior is approached gradually when, with increasing Reynolds number, no stable fixed point or periodic orbit is present and all the unstable ones become more and more unstable, in close analogy with the Lorenz model.

KEY WORDS: Navier–Stokes equations; turbulence; strange attractors; Poincarè map; infinite sequences of periodic orbits; stable and hyperbolic orbits collapse; universal properties in infinite sequences of bifurcations.

1. INTRODUCTION

A model obtained by a suitable five-mode truncation of the Navier–Stokes equations for a two-dimensional incompressible fluid on a torus has been presented in Ref. 1.

The system of nonlinear ordinary differential equations resulting from such a truncation is

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2x_3 + 4x_4x_5\\
\dot{x}_2 &= -9x_2 + 3x_1x_3\\
\dot{x}_3 &= -5x_3 - 7x_1x_2 + r\\
\dot{x}_4 &= -5x_4 - x_1x_5\\
\dot{x}_5 &= -x_5 - 3x_1x_4
\end{aligned}$$

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(where r is the Reynolds number), and exhibits an interesting variety of different behaviors for different ranges of r. Keeping the same symbols as in Ref. 1 for the critical values of r, the most interesting feature is the stochastic behavior observed when $R_{12} < r < R_{13}$, with $28.73 < R_{12} < 29.0$ and $R_{13} \approx 33.43$.

In recent years much attention has been devoted to the study of models exhibiting such a feature when one or more parameters increase beyond certain critical values. The best known models of this kind are certainly the ones by Lorenz[(2-4)] and Hénon.[(5,6)] Ruelle and Takens[(7)] explain this stochastic behavior as a consequence of the appearance of an attractor with a complicated nature ("strange attractor"), on which the motion seems completely chaotic ("turbulence"). In addition to detailed studies on the nature of these attractors (see, for example, Lanford,[(8)] and Hénon and Pomeau[(6)]), strong interest has been focused upon the study of the mechanism of their generation.

In Ref. 1 it is shown that turbulence is reached through a long and rather complicated sequence of bifurcations related to two sequences of orbits: the former consists of four orbits with periods T, $2T$, $4T$, and $8T$, respectively, and the latter of five orbits of a different type with periods T^*, $2T^*$, $4T^*$, $8T^*$, and $16T^*$. In the following, $\mathscr{C}_i$, $i = 0, 1, 2, 3$, will refer to the orbits of the former sequence and $\mathscr{C}_i^*$, $i = 0, 1, \ldots, 4$, to the orbits of the latter.[3] The orbit $\mathscr{C}_0^*$ is found for a value of r larger than but very close to the largest value for which $\mathscr{C}_3$ is still found.[4] It is then suggested that the sequence $\mathscr{C}_i$ is finite, different from $\mathscr{C}_i^*$, and $\mathscr{C}_3$ bifurcates in $\mathscr{C}_0^*$. Since this transition remains an obscure point, because it does not fit very well with the ideas of bifurcation theory, it seems interesting to us to investigate more deeply the two sequences of orbits.

A further reason for this investigation is to verify if the behavior of the sequence $\mathscr{C}_i^*$ is compatible with the strongly suggestive idea of universality in certain infinite sequences of bifurcations developed by Feigenbaum.[(9)] This exhaustive study has been possible because we have been able to apply numerical schemes that are more efficient for studying the stability of orbits and especially in searching for new orbits, even unstable ones. With these new techniques, for a better understanding of the generation of

[3] More precisely, $\mathscr{C}_i$ must be regarded as one of four symmetrically placed, identical orbits going through identical behavior, and the same for $\mathscr{C}_i^*$. There are then four sequences $\mathscr{C}_i$ and four $\mathscr{C}_i^*$, although we will refer simply to "the" sequence $\mathscr{C}_i$ or $\mathscr{C}_i^*$, when not otherwise required. The presence of quadruples of periodic solutions is accounted for by the symmetries $(x_1, x_2, x_3, -x_4, -x_5) \leftrightarrow (x_1, x_2, x_3, x_4, x_5)$, $(-x_1, -x_2, x_3, -x_4, x_5) \leftrightarrow (x_1, x_2, x_3, x_4, x_5)$, $(-x_1, -x_2, x_3, x_4, -x_5) \leftrightarrow (x_1, x_2, x_3, x_4, x_5)$.

[4] $\mathscr{C}_3$ is observed up to $r = 28.6660$, while $\mathscr{C}_0^*$ is first found for $r = 28.6662$.

the "strange attractor" we also have been able to reconsider the transitions through R_{12}, for r increasing, and R_{13}, for r decreasing, with which the system goes over to turbulent behavior. The results of these investigations are described in the following.

In Section 2 a detailed analysis of the two sequences of orbits $\mathscr{C}_i$ and $\mathscr{C}_i^*$ is given, showing that both of them are very likely to be infinite, with a phenomenon of hysteresis because of the simultaneous presence of the orbits $\mathscr{C}_i$, $i \geqslant 3$, with $\mathscr{C}_0^*$.

In Section 3 it is shown that the appearance of the "strange attractor" for r decreasing to R_{13} follows the collapse of the stable orbit present for the high-r regime[5] with an unstable hyperbolic one and that analogous phenomenology is present in the appearance of $\mathscr{C}_0^*$. An interpretation using the theory of the bifurcation of periodic orbits in generic conditions[(7)] is tried.

In Section 4 the compatibility of the, now two, infinite sequences of bifurcations with the universality theory developed by Feigenbaum is verified and some considerations on their asymptotic behavior are inferred.

In Section 5 a detailed analysis of the Poincaré map shows how the turbulent behavior is approached gradually when the previously stable periodic orbits, now all unstable, become more and more unstable for r increasing.

Finally, a schematic picture of the features exhibited by the system is presented in Section 6, together with some concluding remarks.

2. TWO INFINITE SEQUENCES OF BIFURCATIONS

In Ref. 1 it is shown that for a certain value of the Reynolds number r ($r = R_3 = 22.85370163 \cdots$) four previously stable fixed points become unstable and four stable periodic orbits, referred to as $\mathscr{C}_0$ in Section 1, arise via a Hopf bifurcation[6] around each fixed point. With increasing r, the periodic orbits are shown to go through a number of bifurcations, doubling in period and winding up twice as many times around the fixed points from which they are generated. This is shown to happen up to $r = 28.6660$, when three successive bifurcations have taken place, giving rise to the orbits $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$. For $r = 28.6662$ four new stable orbits $\mathscr{C}_0^*$ are found, with structure and period different from the previous ones of $\mathscr{C}_3$, each of them winding up around two of the fixed points. It is stressed that no definite statement can be made about the fact that no further similar bifurcation

[5] We recall that for $r \to R_{13}$ from above, this stable periodic orbit bifurcates with a real eigenvalue crossing the unit circle at $+1$.

[6] For a detailed theory concerning the Hopf bifurcation and its applications see Ref. 11.

takes place in the sequence $\mathscr{C}_i$; all the phenomenology observed, however, leads to the conjecture that $\mathscr{C}_3$ bifurcates in $\mathscr{C}_0{}^*$.

The difficulty in interpreting this point according to the bifurcation theory motivated us to reconsider it, applying Newton's method to obtain and analyze periodic solutions. Once the approximate initial point is close enough, the convergence of the method is fast, both for stable and unstable periodic orbits. The main purpose of such a method is in fact to be able to find unstable periodic solutions too.

Going back to the study of the first sequence of orbits, we have been able to determine the bifurcation points with a very good accuracy and, much more important, to find two more orbits in the sequence $\mathscr{C}_i$, i.e., $\mathscr{C}_4$ and $\mathscr{C}_5$ (see Table I). We have also verified that each of the orbits in the sequence $\mathscr{C}_i$ becomes unstable when an eigenvalue of the Liapunov matrix of the Poincaré map crosses the unit circle at the point -1.[7] The agreement with what is predicted by the bifurcation theory in this case is now complete, since upon bifurcation the previously stable orbit becomes unstable and a new stable orbit appears, with the period doubled (see Ref. 7). At this point it is reasonable to infer that the sequence $\mathscr{C}_i$ is infinite too. We have not carried out a further investigation for higher bifurcated orbits because of the high amount of computational time required even to consider only the next one.

Also for the sequence of orbits $\mathscr{C}_i{}^*$ the bifurcation points have been determined very accurately using the Newton method, up to the orbit $\mathscr{C}_4{}^*$ (see Table II). It has been verified that $\mathscr{C}_i{}^*$ is generated by a sequence of bifurcations of the same kind as those for $\mathscr{C}_i$, since also each orbit $\mathscr{C}_i{}^*$ becomes unstable with an eigenvalue of -1 in the Liapunov matrix of the Poincaré map.

[7] This was already seen in Ref. 1 for the orbit $\mathscr{C}_0$.

Table I. Bifurcation points ρ_i of the Periodic Orbits in the Sequence $\mathscr{C}_i$ and Relative Periods $T(\rho_i)$

i	ρ_i	$T(\rho_i)$
0	28.4105	0.81621
1	28.6399	1.64567
2	28.6641	3.29334
3	28.66776	6.58741
4	28.668463	13.17507
5	28.668611	26.35026

Table II. Bifurcation points ρ_i^* of the Periodic Orbits in the Sequence $\mathscr{C}_i^*$ and Relative Periods $T^*(\rho_i^*)$

i	ρ_i^*	$T^*(\rho_i^*)$
0	28.7013	3.80928
1	28.71606	7.62056
2	28.71926	15.24271
3	28.719947	30.48597
4	28.720103	60.97222

A question left open is how the orbit $\mathscr{C}_0^*$ appears. The answer is that an unstable orbit exists simultaneously with $\mathscr{C}_0^*$, very close to it, and the two collapse upon bifurcation for r decreasing. All the details will be given in the next section.

An important feature must be emphasized concerning the two sequences of orbits $\mathscr{C}_i$ and $\mathscr{C}_i^*$, i.e., the simultaneous presence of different stable orbits for the same value of the Reynolds number r in a certain range of r. For $r \geqslant 28.663$ in fact the stable orbit $\mathscr{C}_0^*$ is present together with one of the sequence $\mathscr{C}_i$. It is clear that $\mathscr{C}_0^*$ appears in the beginning with a very small basin of attraction; with increasing r, this becomes larger and larger, while that of the simultaneous orbit $\mathscr{C}_i$ gets smaller and smaller. $\mathscr{C}_4$ already appears with a very small basin of attraction and $\mathscr{C}_5$ even more so. The simultaneous presence of more than one attracting orbit is termed hysteresis and has the effect of causing a rather sensitive dependence of the asymptotic solution on the initial conditions. In this kind of model this was found by Curry[12] in a generalized Lorenz system and by us[13] as a very strong feature in a seven-mode truncation model of the two-dimensional Navier–Stokes equations.

3. COLLAPSE OF A STABLE ORBIT WITH AN UNSTABLE ONE

One of the interesting results of Ref. 1 is the observation that, after the second infinite sequence of orbits, the system shows the presence of two symmetric attractors, with all the characteristics of a "strange attractor," on which a random motion takes place. With increasing Reynolds number r, each attractor seems to shrink to a stable periodic orbit, present for all the high-r regime considered. Analysis of the stability of this orbit shows that, with decreasing r toward $R_{13} = 33.43$, an eigenvalue of the Poincaré map approaches the unit circle at the point $+1$. It seems of interest to us to attempt to obtain a better understanding of the way this bifurcation takes place, as it is connected with the transition to turbulence.

We have verified that after the bifurcation, i.e., for $r < R_{13}$, the orbit is no longer present. This fact suggests the hypothesis that the stable orbit could disappear by collapse with an unstable one present at the same time (see, for example, Brunowsky[14]). We have looked at how the fixed point of the Poincaré map for the stable orbit, using a fixed hyperplane, would move when r decreases toward R_{13}. Keeping in mind the hypothesis of collapse, by extrapolation we have been able to find the fixed point for an unstable orbit at a value of r close to the critical one. We have then followed the unstable orbit present together with the stable one, quite close to it, and disappearing for $r < R_{13}$. It has been verified that, with decreasing r toward R_{13}, the two orbits become closer and closer (see Fig. 1) and so do their periods, that for the stable orbit increasing, that of the unstable one decreasing. Figure 2, where the fixed points of the Poincaré map for the stable and unstable orbits are represented for different values of r approaching R_{13} from above, shows the phenomenon of collapse quite clearly.

The same detailed analysis performed on the stable orbit present in the high-r regime has been carried on for $\mathscr{C}_0{}^*$, since a study of its stability shows an eigenvalue of the Poincaré map approaching the unit circle at $+1$ for r

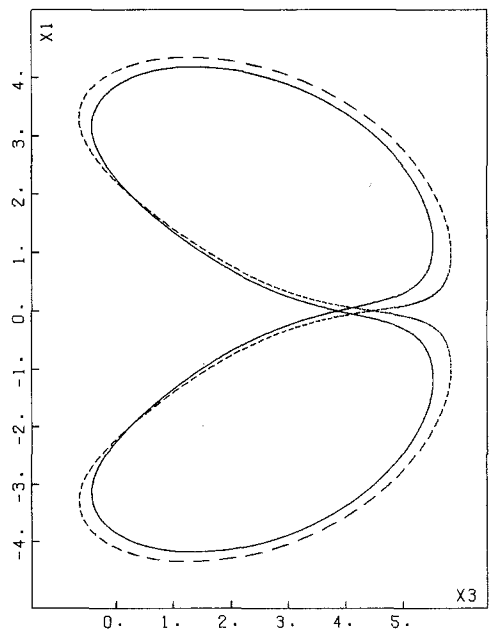


Fig. 1. Stable (——) and hyperbolic (- - -) orbits for $r = 33.60$.

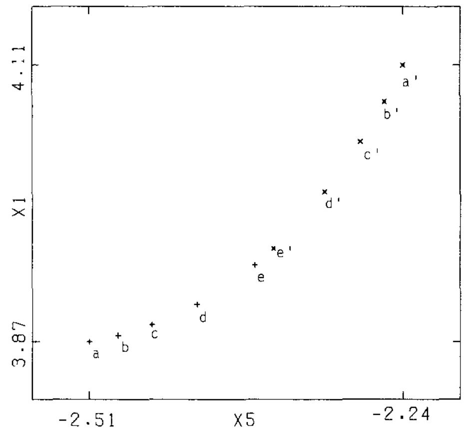


Fig. 2. Fixed points of the Poincaré map for the stable (+) and hyperbolic (×) orbits for r approaching R_{13} from above: (a, a') $r = 33.80$; (b, b') $r = 33.70$; (c, c') $r = 33.60$; (d, d') $r = 33.50$; (e, e') $r = 33.44$.

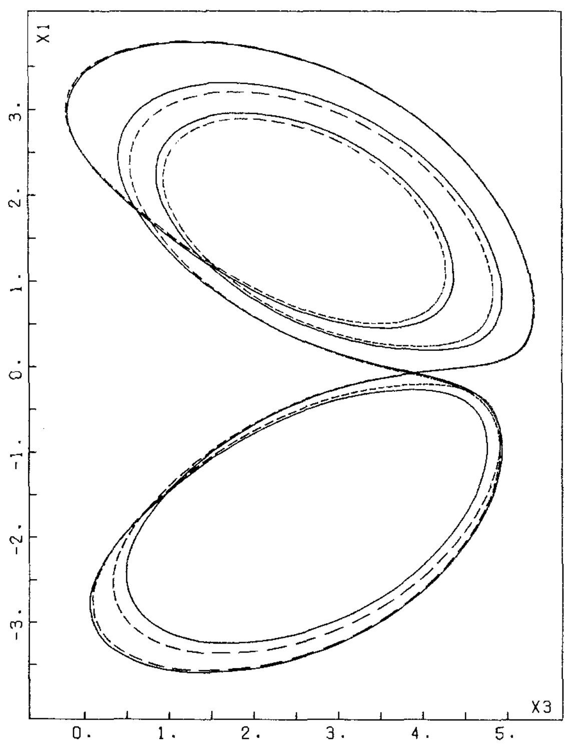
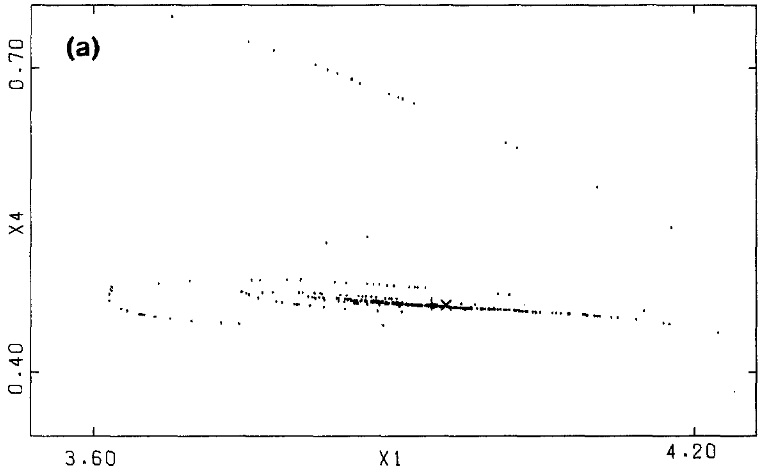


Fig. 3. Stable (——) and hyperbolic (- - -) orbits for $r = 28.695$.



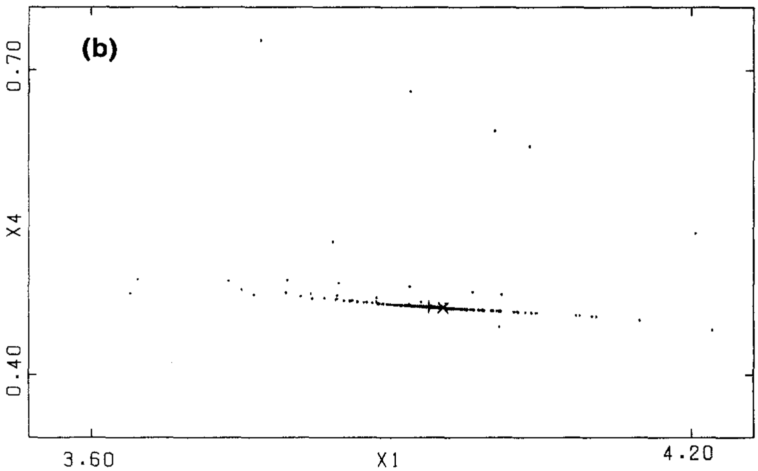


Fig. 4. Projections of the Poincaré map on the plane (x_1, x_4) for r = (a) 33.300; (b) 33.430; (c) 33.4385; (d) 33.440. The symbol + (×) represents the fixed point of the stable (hyperbolic) orbit for r = 33.440.

decreasing toward 28.663, the orbit disappearing below that value. For $\mathscr{C}_0{}^*$ too we have verified the identical phenomenon of collapse with an unstable orbit $\bar{\mathscr{C}}_0{}^*$ (see Fig. 3); the difference from the previous case is that no attractor close to the orbit is present after the bifurcation. The presence of the orbit $\bar{\mathscr{C}}_0{}^*$, besides explaining the bifurcation for r = 28.663, plays a role in the explanation of how the model goes over to "turbulent" behavior, as will be seen in Section 5.

In their fundamental paper, among other considerations, Ruelle and

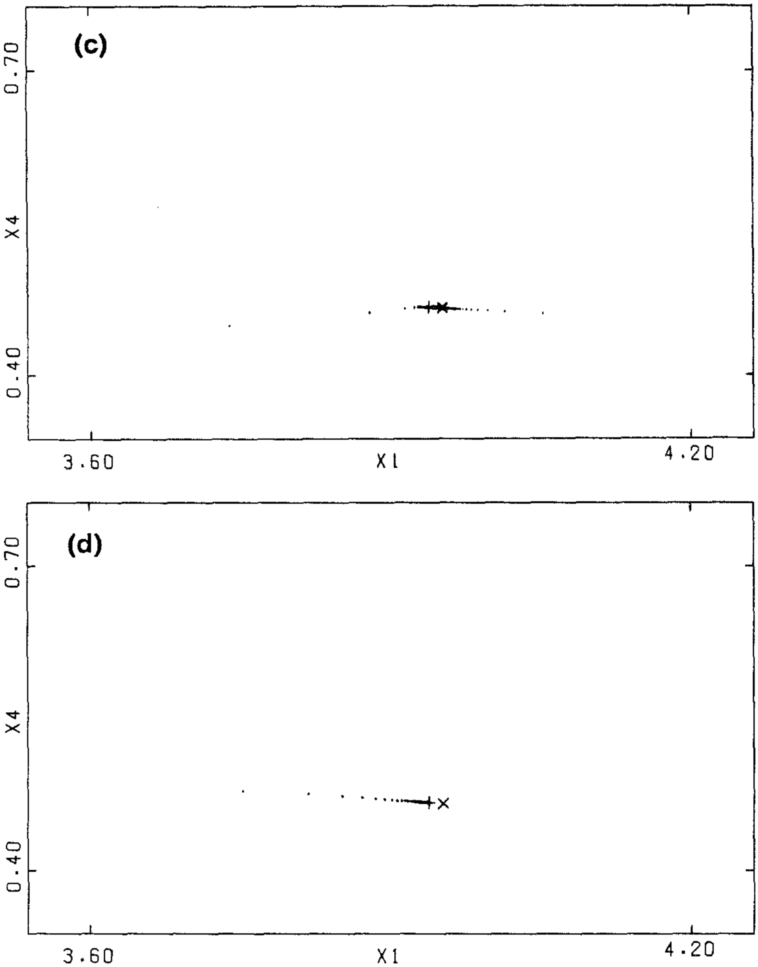


Fig. 4. Continued.

Takens[7] analyze the bifurcations of a stable periodic orbit in generic systems when the Poincaré map has only a finite number of isolated eigenvalues with modulus 1 and the others inside the open unit circle. They find that when only one eigenvalue crosses the unit circle at the point +1, one should expect the attracting closed orbit to disappear together with a hyperbolic closed one and no attractor close to the orbit to appear after the bifurcation has taken place. This is the exact phenomenology found for $\mathscr{C}_0{}^*$, but at first sight not that for the orbit in the high-r regime. In this last case in fact at the bifurcation point we observe the collapse of the two orbits, but after the bifurcation a strange attractor is present, apparently close to the orbits.

This phenomenology does not seem to be in agreement with Ref. 7, so we have tried to study in more detail the transition from the strange attractor to the periodic orbits.

For different values of r approaching R_{13} from below, we have considered the Poincaré map on the hyperplane $x_3 = 3.0$, $x_1 \geqslant 0$, plotting its projection on (x_1, x_4) together with, as a reference, the fixed points of the stable and hyperbolic orbits for r slightly greater than R_{13} ($r = 33.44$).

Figures 4a–4c clearly show how, as r approaches R_{13}, the points tend to dispose themselves along a line, getting denser and denser on a segment containing the fixed points of the two orbits. Looking at the projection of the intersection points on the plane (x_1, x_4), this segment is described from left to right, with a return mechanism which redescribes it always progressing in the same direction. Moreover, numerical evidence has been found for the two following facts: for $r \rightarrow R_{13}$, $33.4385 < R_{13} < 33.4390$, the length of the segment does not seem to tend to zero; at $r = R_{13}$ a stable fixed point for the Poincaré map appears on the segment.

A possible qualitative explanation for the phenomenology is the existence of some attracting variety, containing the "strange attractor" for $r < R_{13}$, and a stable periodic orbit, together with an unstable one, for $r > R_{13}$. The presence of such a manifold, attracting also for $r > R_{13}$, appears to be confirmed in Fig. 4d, where the approach to the fixed point of the stable orbit on the Poincaré map is shown for $r = 33.44$. Looking at the phenomenon for r increasing, we have then that the strange attractor does not "shrink" to the orbit, in agreement with the bifurcation theory.

For r decreasing, the stable orbit and the hyperbolic one collapse and, disappearing, are replaced by a "larger" attractor which occupies a portion of a manifold to which the orbits are always attracted for r near R_{13}, no matter whether larger or smaller; the diameter of the attractor does not tend to zero as $r \rightarrow R_{13}$.

4. COMPATIBILITY WITH A CONJECTURE OF UNIVERSALITY IN INFINITE SEQUENCES

In a recent paper, Feigenbaum[9] develops a very interesting theory concerning a large class of recursion relations $x_{n+1} = \lambda f(x_n)$ exhibiting infinite bifurcations, varying the parameter λ in the open interval $(0, 1)$. They are shown to possess a structure essentially independent of the recursion function that, among other properties, is supposed to map the closed interval $[0, 1]$ on itself and have a unique, twice differentiable maximum $\bar{x}$. For such a class of f, a λ_n exists such that a stable 2^n-point limit cycle including $\bar{x}$ exists. It is shown as numerical evidence that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta$$

i.e., the λ_n geometrically converge to a certain λ_∞ at the rate δ, independent of the specific function f.

The same asymptotic behavior is shown to take place for

$$\frac{\Lambda_n - \Lambda_{n-1}}{\Lambda_{n+1} - \Lambda_n}$$

where Λ_n is the nth bifurcation point. Moreover, when λ is increased in order to obtain the transition from a stable 2^n-point to a stable 2^{n+1}-point limit cycle, the local structure about $\bar{x}$ reproduces itself on a scale α_n times smaller. It is shown that

$$\lim_{n \to \infty} \alpha_n = \alpha$$

with α also f-independent. Both the numbers α and δ depend only on the order of the maximum $\bar{x}$ of f; for a normal (i.e., quadratic) maximum, it is found that $\delta = 4.6692 \cdots$ and $\alpha = 2.5029 \cdots$.

In a previous section we have described two sequences of bifurcations that are very likely to be infinite, even if for the reasons exposed there we could obtain only a limited number of terms. We have tried to verify numerically if the universal metric properties pointed out by Feigenbaum for one-dimensional mappings could hold in our dynamical system too, hoping for a convergence of α_n and δ_n as fast as that of one of the examples in Ref. 9. We have computed the ratios $\delta_i = (\rho_i - \rho_{i-1})/(\rho_{i+1} - \rho_i)$, $i = 1,..., 5$, for the sequence $\mathscr{C}_i$ and $\delta_i^* = (\rho_i^* - \rho_{i-1}^*)/(\rho_{i+1}^* - \rho_i^*)$ for $\mathscr{C}_i^*$, where ρ_i and ρ_i^* are the bifurcation points given in Tables I and II. Both sequences δ_i and δ_i^*, listed in Table III, seem to indicate a convergence, more rapid for δ_i^*, to numbers quite compatible with the one found by Feigenbaum. A comment is required by the last term δ_4^*. This has been computed knowing quite well that it might not have been completely reliable. In fact the numerical errors due to the large value of the period of $\mathscr{C}_4^*$ now become relevant compared with the very small variation of the parameter r. We have computed the term anyway because of the small number of terms available otherwise, to verify at least a persistence of the sequence around the value of δ.

Table III

i	δ_i	δ_i^*
1	24.22	2.57
2	9.48	4.63
3	6.54	4.64
4	5.29	4.42
5	4.73	—

An even more striking instance of the compatibility of our sequences with the one considered by Feigenbaum is found by looking at how the fixed points of the Poincaré map reproduce themselves upon transition from each orbit to the next one in the sequence. Calling Φ_r the Poincaré map on a hyperplane transverse to each orbit $\mathscr{C}_i$, we can write a recursion relation

$$x_{n+1} = \Phi_r(x_n)$$

When, for $\rho_{i-1} < r < \rho_i$, we consider the stable orbit $\mathscr{C}_i$, Φ_r has a stable 2^i-point limit cycle. In this way we have a recursion function similar to the one in Ref. 9 since in the ith bifurcation point ρ_i a 2^i-point limit cycle becomes unstable and a stable 2^{i+1}-point limit cycle appears, with the stable orbit $\mathscr{C}_{i+1}$ appearing.

A comparison with Feigenbaum's scale factors α_n can be attempted once something corresponding to $\bar{x}$ is found. Considering our limit cycles at the bifurcation points, we have observed the following. Calling $P_j^{(i)}$, $i = 0, 1,..., 5$, $j = 1,..., 2^i$, the points of the 2^i-point cycle, and $P_{2j-1}^{(i+1)}$ and $P_{2j}^{(i+1)}$ the two points bifurcating from $P_j^{(i)}$ in the 2^{i+1}-point cycle, we let

$$Q_1^{(i)} = P_{2k_0-1}^{(i)}, \qquad Q_2^{(i)} = P_{2k_0}^{(i)}$$

where k_0 is the index for which $d(P_{2k-1}^{(i)}, P_{2k}^{(i)})$ is maximum,[8] $k = 1,..., 2^{i-1}$. Then, for $i \geqslant 2$, we have that $Q_1^{(i)}$ and $Q_2^{(i)}$ correspond either to $Q_1^{(i-1)}$ or to $Q_2^{(i-1)}$. This is equivalent to saying that if we consider the binary tree with the points $P_j^{(i)}$ as nodes of level $i + 1$, a path from $P_1^{(0)}$ to a certain $P_j^{(5)}$ exists, along which the points $P_j^{(i)}$ reproduce themselves with a scale factor that is maximum.

In Ref. 9 the scale factor, by definition $1/\alpha_i$, by which a cluster about a point of a 2^i-cycle reproduces itself is maximum if the point is $\bar{x}$. For this reason it seems relevant to compute the α_i along the path we have specified before, proposing in this way a correspondence between $Q^{(i)}$ and $\bar{x}$. In Table IV we list the values of the α_i for the Poincaré map on the hyperplane $x_3 = 1.0$ for each coordinate and the Euclidean distance.

The same procedure has been followed for the sequence $\mathscr{C}_i^*$, the corre-

[8] d is the usual Euclidean distance in R^5.

Table IV

	x_1	x_2	x_4	x_5	d
α_1	4.20	4.21	5.05	5.09	4.51
α_2	3.43	3.43	3.29	3.30	3.39
α_3	2.90	2.90	2.94	2.94	2.91
α_4	2.53	2.53	2.52	2.52	2.53

Table V

	x_1	x_2	x_4	x_5	d
$\alpha_1{}^*$	2.31	2.31	2.38	2.38	2.34
$\alpha_2{}^*$	2.65	2.65	2.60	2.60	2.63
$\alpha_3{}^*$	2.43	2.43	2.45	2.45	2.44

sponding limit cycle being $m \cdot 2^i$-point, where m is dependent on the plane chosen for the Poincaré map. The scale factors $\alpha_i{}^*$ for this case are given in Table V, also for the Poincaré map on $x_3 = 1.0$, limited to $x_1 > 0$, with $m = 3$.

For different choices of the hyperplane for the Poincaré map the results are essentially unchanged.

The compatibility of our numerical values for α_i and $\alpha_i{}^*$ with the asymptotic value of α computed by Feigenbaum seems evident, even if we could compute only a few terms.

Our results make possible the hypothesis that universal metric properties of one-dimensional mappings also hold in dynamical systems with infinite sequences of bifurcations. The fact that complicated n-dimensional phenomena possess characteristics in some sense "one-dimensional" appears significant and very suggestive.[9]

These arguments give more support to our hypothesis of Section 2 on the two sequences being infinite and allow us now to estimate the asymptotic values for the critical Reynolds number ρ_i and $\rho_i{}^*$. We obtain for them

$$\rho_\infty = 28.668652, \qquad \rho_\infty{}^* = 28.720135$$

A complete numerical definition of these values from the model appears impossible, however.

5. ONSET OF TURBULENCE

In Ref. 1 it is shown that for $R_{12} < r < R_{13}$, randomly chosen initial data lead to two attractors (see Fig. 5), on which the motion appears to be completely random, the trajectories looking exactly like the ones found by Lorenz in his model.[2] These two "strange attractors" are localized in two

[9] A detailed study generalizing Feigenbaum's results has been carried out by Derrida *et al.*[10] Moreover, they have pointed out that also in the Hénon two-dimensional mapping[5] the bifurcation rate δ for the sequence of stable periods 2^n is the same as in Ref. 9. An intuitive explanation for this is indicated by the contracting nature of the transformation.

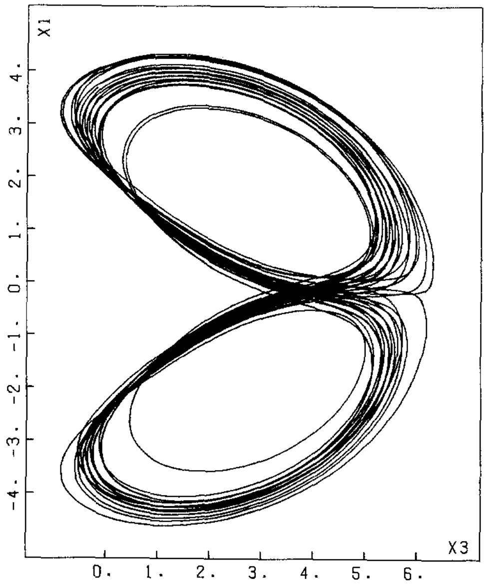


Fig. 5. One of the two "strange attractors" for $r = 33.0$.

symmetric regions, each of them surrounding two fixed points, two sequences of orbits $\mathscr{C}_i$, two sequences $\mathscr{C}_i^*$,[10] and two orbits $\overline{\mathscr{C}}_0^*$, all unstable in this range of the Reynolds number r.

In the following we give a detailed analysis of the transition from the periodic behavior of the sequences $\mathscr{C}_i^*$ to the turbulent behavior on the two attractors.

We consider the motion with random initial data for different values of r, starting from $r = 28.72$, studying the Poincaré map on the hyperplane $x_3 = 1.4$, limited to the region $x_1 \geqslant 0$, $x_4 \geqslant 0$ for simplicity.[11] The orbits of only one of the sequences $\mathscr{C}_i$, the orbits of two of the sequences $\mathscr{C}_i^*$, and two orbits $\overline{\mathscr{C}}_0^*$, all intersecting the hyperplane $x_3 = 1.4$, are present in the region considered. These intersections are the elements of n-point limit cycles for the Poincaré map. Denoting by c_i a 2^i-point limit cycle related to an orbit $\mathscr{C}_i$, and by $c_{m,i}^*$ an $m \cdot 2^i$-point cycle related to an orbit $\mathscr{C}_i^*$, we find the Poincaré map then has one sequence of cycles c_i, one $c_{2,i}^*$, one $c_{3,i}^*$, one cycle $\bar{c}_{2,0}^*$, and one $\bar{c}_{3,0}^*$. The complexity of the situation is evident from Fig. 6a, where for $r = 28.72$ we have represented only the cycle $c_{3,4}^*$, stable for this value of r, and the unstable cycles c_0, c_1, $c_{2,0}^*$, $c_{3,0}^*$, $\bar{c}_{2,0}^*$, $\bar{c}_{3,0}^*$. The

[10] For a better understanding see Ref. 1, especially Figs. 1a, 7a–7d, and 9–11.

[11] Suitable changes in sign, allowed for by the symmetries present in the model, make it possible to study any orbit in this region.

points of the stable cycle $c_{3,4}^*$ accumulate in three groups in a neighborhood of the three points of the cycle $c_{3,0}^*$ from which they have bifurcated. Because of the scale factor chosen in order to represent a complete picture of the Poincaré map, the points in each group appear to be practically indistinguishable, but show in the plane (x_1, x_4) a line along which they duplicate at each bifurcation. This fact is clearly evident in Fig. 6a′, where on an enlarged scale the central group of the points of the stable cycle $c_{3,4}^*$ is represented together with the points of the cycles $c_{3,0}^*$ and $c_{3,1}^*$, now unstable, from which they have bifurcated: they also appear on the line of the points of $c_{3,4}^*$. Even if we give no evidence for this, because of the high computational time required, it is reasonable to think that the points of $c_{3,2}^*$ and $c_{3,3}^*$ also stay on the same line.

A natural extension of this argument is the hypothesis that also for $r > \rho_\infty{}^*$ all the points of the full sequence $c_{3,i}^*$ are disposed in an analogous way along the same line. The same study carried on for the sequence c_i shows an analogous phenomenology.

Let us examine now the behavior of the flow for r slightly larger than 28.72. Figures 6*b*–6d show the projections on the plane (x_1, x_4) of the hyperplane chosen for the Poincaré map for $r = 28.721$, $r = 28.723$, and $r = 28.730$, respectively. In all three figures we see the results of 400 intersections of the solution curve with our codimension-one section. It is observed that the behavior of the flow is still very much analogous to that observed for $r = 28.72$ when the stable orbit $\mathscr{C}_4{}^*$ is present. The numerical data obtained do not allow us at all to state whether the observed motion is periodic, possibly with a very long period, or not. The figures show, however, that with increasing r the intersection points keep disposing themselves only on arcs along the direction identified by the points of the cycle $c_{3,4}^*$, now unstable, but become more and more spread, and they tend to approach the points of the cycle $\bar{c}_{3,0}^*$ (see Figs. 6a–6d). An examination of the values of the coordinates of all these points seems to show more and more randomness with increasing r, confirmed by Fig. 6e, where we see that the behavior of the flow for $r = 28.732$ is definitely changed. In fact it is possible to observe that now also the points of the cycles $c_{2,i}^*$ due to the second sequence $\mathscr{C}_i{}^*$ and of the cycle $\bar{c}_{2,0}^*$ due to the second orbit $\bar{\mathscr{C}}_0{}^*$ contribute to the behavior of the solution curve. Moreover, some intersection points now seem to be arranged rather randomly and not to be connected with any one of the unstable cycles of the Poincaré map. With continued increasing r, we observe a more and more chaotic behavior, due to a gradual involvement of the orbits $\mathscr{C}_i$ also, since the intersection points are now also close to the fixed points of the orbits $\mathscr{C}_0$ and $\mathscr{C}_1$. Figure 6f shows the motion for $r = 28.80$, appearing fully random around the points of the unstable n-cycles present in the region.

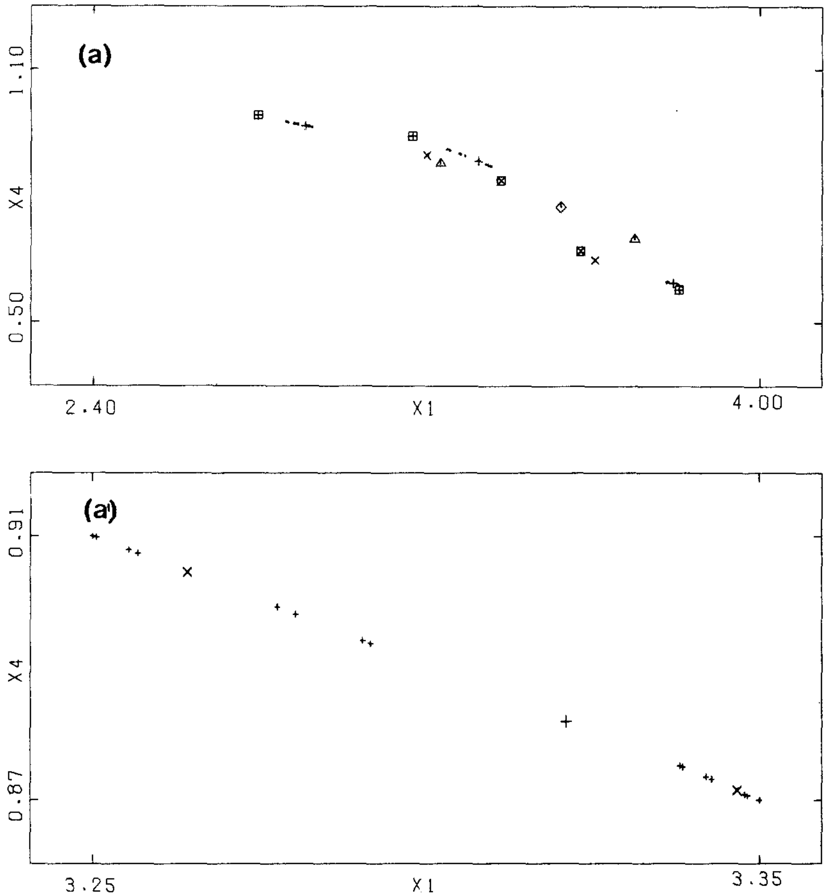


Fig. 6. Projections of the Poincaré map on the plane (x_1, x_4) for r = (a) 28.720; (b) 28.721; (c) 28.723; (d) 28.730; (e) 28.732; (f) 28.800. (+) the three points of the cycle $c_{3,0}^*$; (×) the two points of the cycle $c_{2,0}^*$; (⊞) the three points of the cycle $\bar{c}_{3,0}^*$; (⊠) the two points of the cycle $\bar{c}_{2,0}^*$; (◇) the fixed point of c_0; (△) the two points of the cycle c_1. (a′) Points of the central group [part (a)] of the stable cycle $c_{3,4}^*$ (+) on an enlarged scale, with the points of $c_{3,0}^*$ (+) and $c_{3,1}^*$ (×) from which they have bifurcated.

The phenomenology described above does not allow a rigorous definition of the mechanism of the onset of turbulence. In fact we are unable to evaluate $\rho_\infty{}^*$ exactly, i.e., the critical value of the Reynolds number r for which the sequence $\mathscr{C}_i{}^*$ exhausts itself, and we cannot state definitely what happens for r slightly greater than $\rho_\infty{}^*$. The existence of more infinite sequences of stable periodic orbits in very small ranges of r or with very long periods then cannot be rejected.

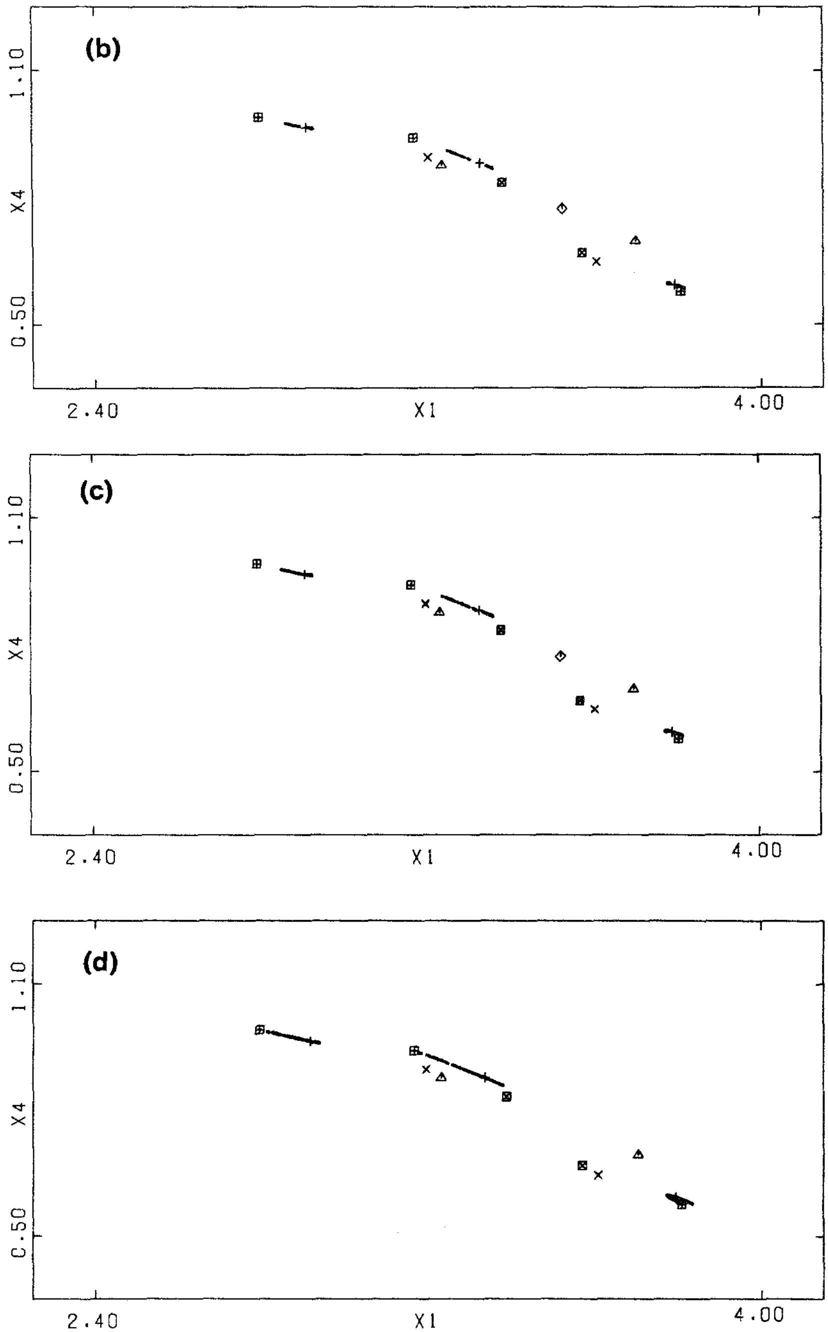


Fig. 6. Continued.

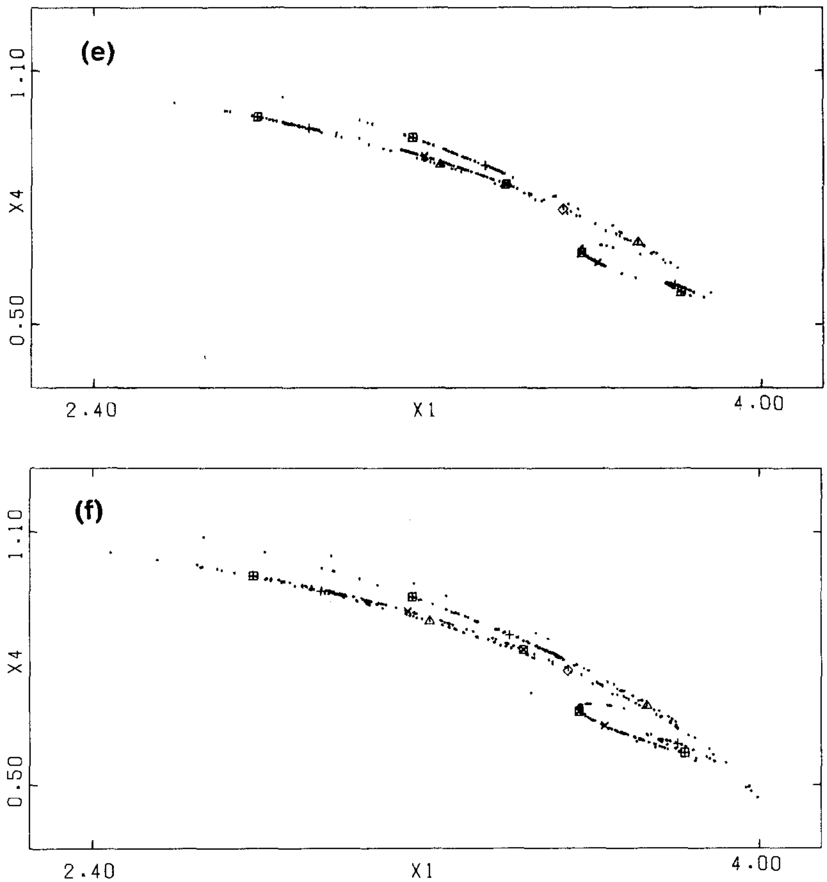


Fig. 6. Continued.

We think, however, that we can interpret the numerical results in the following way. The characteristics of the sequence $\mathscr{C}_i^*$ indicate that it tends rapidly to exhaust itself when r approaches a value very likely to be quite close to ρ_∞^*, computed in the previous section according to Feigenbaum's theory. For $r > \rho_\infty^*$, when all the orbits have become unstable, the system possesses only unstable fixed points and periodic orbits: the seven fixed points (see Ref. 1), the four sequences of orbits $\mathscr{C}_i$, the four sequences $\mathscr{C}_i^*$, and the four orbits $\bar{\mathscr{C}}_0^*$. Up to $r \cong 28.73$ any random initial value is attracted by the orbits of one of the four sequences $\mathscr{C}_i^*$, which are definitely unstable, but being less unstable than the others, succeed in "catching" the point and keep it trapped in their neighborhood, at least for the long time intervals observed. With increasing r, the orbits $\mathscr{C}_i^*$ become more unstable and gradually lose the ability to keep the point trapped, allowing it also to approach the other unstable periodic orbits. For r further increased all the orbits have lost more and more stability and the point "jumps" more easily from the neighborhood of an orbit to that of another. The motion then becomes more and more chaotic, even remaining confined in one of the two distinct symmetric regions where the unstable orbits are localized.

The two strange attractors in our system then appear as a consequence of the instability of all the orbits present, i.e., with a mechanism perfectly analogous to the one in the "standard" Lorenz attractor,[3] even if much more complicated.

6. CONCLUSION

Because of the complicated phenomenology present in the considered model of the five-mode truncated Navier–Stokes equations, it seems useful to present a schematic picture of the features found. Redefining the sequence of the critical values of r with $R_1' = R_3$, $R_2' = 28.663$, $R_3' = \rho_\infty$, $R_4' = \rho_\infty^*$, and $R_5' = R_{13}$, we have:

(a) For $0 < r \leqslant R_1'$ the model exhibits only stationary solutions (see Ref. 1 for details).

(b) For $R_1' < r < R_3'$ the system, through an infinite sequence of bifurcations, gives rise to four infinite sequences of symmetric orbits $\mathscr{C}_i$, each one with a period double that of the previous one.

(c) For $R_2' \leqslant r < R_4'$ a further sequence of infinite bifurcations gives rise to four more infinite sequences of orbits $\mathscr{C}_i^*$, also symmetrically placed and with doubled period, but with a more complicated spatial structure.

(d) For $R_4' \leqslant r < R_5'$ all the periodic orbits present in the system are unstable and an erratic, chaotic motion takes place on two symmetric "strange" attractors, analogous to the Lorenz model ("turbulence").

(e) For $r \geqslant R_5'$ two stable periodic orbits are present.

At this point a detailed knowledge of the phenomenology of the model seems to have been reached. In particular we remark the fact that the turbulent behavior is reached gradually when no stable fixed point or periodic orbit is present and all the unstable ones keep losing stability with increasing r. Also a relevant feature is the fact that the two infinite sequences of bifurcations present seem to possess certain characteristics or universality analogous to the ones found by Feigenbaum in nonlinear transformations of an interval in itself.

We conclude by proposing two basic questions: How does a different choice of the five modes for the truncated Navier–Stokes equations effect the behavior of the model, and how does an increase in the number of modes in the truncation affect the model? Concerning the last question, an ongoing study of a seven-mode truncation obtained by adding two more modes to the five used in this study seems to show a rather different phenomenology, with much more variety and strong features of hysteresis.

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